Construction of solvable quantum mechanical potentials

2003 J. Phys. A: Math. Gen. 36 10071

(http://iopscience.iop.org/0305-4470/36/39/307)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 10071–10082 PII: S0305-4470(03)63120-3

Construction of solvable quantum mechanical potentials

S A S Ahmed[1] **and B C Borah**[2]

[1] Department of Physics, Gauhati University, Guwahati 781014, India
[2] Department of Physics, Assam Engineering College, Guwahati 781013, India

Received 7 May 2003, in final form 22 July 2003
Published 17 September 2003
Online at stacks.iop.org/JPhysA/36/10071

Abstract
A method of generating some quantum mechanical exactly solvable potentials (ESPs), using the properties of classical orthogonal polynomials (COPs), is presented. It is illustrated using hypergeometric polynomial. Utilization of other COPs to get more ESPs is indicated.

PACS numbers: 02.90.+p, 03.65.Db, 03.65.Ge

1. Introduction

The potential governing a quantum system (QS) very often does not yield an exact solution of the Schrödinger equation. Therefore, one has to depend on various approximation schemes such as perturbation theory, variational technique, WKB method, etc. However, the success of the application of these approximation schemes in getting the information for a given QS largely depends on the 'nearness' of the potential to some exactly solvable potentials (ESPs). This motivates us to find more and more ESPs which will facilitate practical quantum mechanical calculations. Besides, non-perturbative solutions of different potentials may lead to new physical ideas in quantum physics. This prompted one of the present authors to device a transformation method (Ahmed *et al* 2001) in which one has to start from a known analytically solved QS and transform it to generate new QS(s). Many authors also contributed valuable works in this direction (Manning 1935, Biswas *et al* 1971, Khare 1981, Roy and Roychoudhury 1987, Chhajlany and Malnev 1990, Bose 1994) including the works (Nieto 1984) inspired by supersymmetric (SUSY) quantum mechanics (Cooper *et al* 1995). Besides the normal exactly solvable QSs other analytically solvable QSs include the quasi-exactly solvable QS (Flessas 1979, Flessas and Das 1980, Dutra 1988, Shiffman 1989, Dutra and Filho 1991), conditionally exactly solvable QS (Dutra 1993, Dutta *et al* 1995a) and conditionally quasi-exactly solvable QS (Dutta 1995b). A preliminary version of the present work to generate ESP based on the properties of classical orthogonal polynomials (COPs) in any desired dimensional space has already been published (Ahmed and Borah 2002, henceforth referred to as I).

In this paper, we present a method to construct an exactly solvable quantum system (EQS), which is a refined and expanded version of our earlier report (I). We consider a difactor wavefunction $\psi(r) = F(G(r))Q_n^m(G(r))$, each factor of which is assumed to be a function of another smooth continuous function $G(r)$ which is at least thrice differentiable. The technique is to establish a D-dimensional radial Schrödinger equation for the difactor wavefunction $\psi(r)$. The expression for the differential part $\psi''/\psi + (D-1)r^{-1}\psi'/\psi \equiv \Omega(r)$, say, of the Schrödinger equation $\Omega(r) + (E_n - V(r)) = 0$ is calculated taking $Q_n^m(r)$ to be one of the known COPs in the difactor wavefunction $\psi(r)$. The analytic form of $\Omega(r)$ is then reducible to a multiterm expression containing $G(r)$ and its higher order derivatives using the specific differential equation of the COP considered. $\Omega(r)$ is then cast in the form $(V(r) - E)$ to get the standard form of radial stationary state Schrödinger equation in D-dimensions by selecting one of the terms of $\Omega(r)$ and putting it constant. This last step, which is one of the main ingredients of the method, can be generalized as discussed in the formalism. The constant eventually becomes the energy eigenvalues E_n. The above mentioned step, besides providing E_n, also gives a differential equation in $G(r)$ which yields the functional form of $G(r)$. This in turn allows us to write $\Omega(r)$ in the form $V(r) - E_n$ from which the ESP $V(r)$ can be read off. This enables us to obtain an EQS having a specific potential, energy eigenvalues and normalizable eigenfunctions. The normalizability of the wavefunction gives constraints on the n, m values of the COP, $Q_n^m(G(r))$ and proper choices of these values in conformity with the constraints make the eigenfunction normalizable. However, it should be noted that, the procedure elucidated above gives EQSs which are mostly Sturmian EQSs in nature. By applying system specific conversion techniques as given in the illustrative examples some of them can be converted to normal EQSs, equipped with an 'n-independent' potential and normalizable energy eigenfunctions.

2. Formalism

We consider a difactor radial wavefunction $\psi(r)$ given by

$$\psi(r) = F(G(r))Q_n^m(G(r)) \tag{1}$$

each factor of which is assumed to be a function of another smooth continuous function $G(r)$ at least thrice differentiable. $Q_n^m(r)$ is generic function and will be identified as one of the COPs. The radial Schrödinger equation in D-dimensional Euclidean space is established for the wavefunction $\psi(r)$ and is

$$\begin{aligned}\frac{\psi''(r)}{\psi(r)} + \frac{D-1}{r}\frac{\psi'(r)}{\psi(r)} &= G'^2(r)\frac{F''(G)}{F(G)} + \left\{G''(r) + \frac{D-1}{r}G'(r)\right\}\frac{F'(G)}{F(G)} \\ &\quad + G'^2(r)\frac{Q_n^{m\prime\prime}(G)}{Q_n^m(G)} + \left\{G'' + \frac{D-1}{r}G' + 2G'^2\frac{F'(G)}{F(G)}\right\}\frac{Q_n^{m\prime}(G)}{Q_n^m(G)} \\ &= G'^2(r)\frac{F''(G)}{F(G)} + \left\{G''(r) + \frac{D-1}{r}G'(r)\right\}\frac{F'(G)}{F(G)} \\ &\quad + G'^2(r)\left(\frac{Q_n^{m\prime\prime}(G)}{Q_n^m(G)} + \left(\frac{G''}{G'^2} + \frac{D-1}{rG'} + 2\frac{F'(G)}{F(G)}\right)\frac{Q_n^{m\prime}(G)}{Q_n^m(G)}\right)\end{aligned} \tag{2}$$

where a prime denotes differentiation with respect to its argument. We segregate the $Q_n^m(G)$-dependent part of equation (2) and term it $J_{n,m}$. The form of equation (2) then becomes

$$\frac{\psi''(r)}{\psi(r)} + \frac{D-1}{r}\frac{\psi'(r)}{\psi(r)} = G'^2(r)\frac{F''(G)}{F(G)} + \left(G''(r) + \frac{D-1}{r}G'(r)\right)\frac{F'(G)}{F(G)} + G'^2(r)J_{n,m} \tag{3}$$

where

$$J_{n,m} = \frac{Q_n^{m''}(G)}{Q_n^m(G)} + \left(\frac{G''}{G'^2} + \frac{D-1}{rG'} + 2\frac{F'(G)}{F(G)}\right)\frac{Q_n^{m'}(G)}{Q_n^m(G)}$$
$$= \frac{Q_n^{m''}(G)}{Q_n^m(G)} + \left(\frac{\mathrm{d}}{\mathrm{d}G}\ln\frac{r^{D-1}(G)F^2(G)}{r'(G)}\right)\frac{Q_n^{m'}(G)}{Q_n^m(G)}. \tag{4}$$

The general form of the differential equation of COPs in the independent variable G can be written in the form

$$\frac{Q''(G)}{Q(G)} + M(G)\frac{Q'(G)}{Q(G)} + J_{n,m} = 0. \tag{5}$$

Consistency then demands that $\left(\frac{\mathrm{d}}{\mathrm{d}G}\ln\frac{r^{D-1}(G)F^2(G)}{r'(G)}\right)$ in equation (4) must be identified as the $M(G)$ of equation (5). This fixes the form of $F(G)$ for a given COP as

$$F(G) = NG'^{-\frac{1}{2}}r^{-\frac{D-1}{2}}\left(\exp\left(\int M(G)\,\mathrm{d}G\right)\right)^{\frac{1}{2}} \tag{6}$$

where N is the integration constant and plays the role of normalization constant of the energy eigenfunctions.

Equation (6) gives

$$\frac{F'(G)}{F(G)} = -\frac{D-1}{2rG'} + \frac{1}{2}M(G) - \frac{1}{2}\frac{G''}{G'^2} \tag{7}$$

and

$$\frac{F''(G)}{F(G)} = \frac{(D-1)^2}{2r^2G'^2} - \frac{G'''}{2G'^3} + \frac{5}{4}\frac{G''^2}{G'^4} + \frac{(D-1)G''}{rG'^3} + \frac{1}{4}(M(G))^2$$
$$- \frac{(D-1)}{2rG'}M(G) + \frac{1}{2}M'(G) - \frac{1}{2}\frac{G''}{G'^2}M(G). \tag{8}$$

Equations (3), (7) and (8) yield

$$\frac{\psi''(r)}{\psi(r)} + \frac{D-1}{r}\frac{\psi'(r)}{\psi(r)} = -\frac{1}{2}\{G,r\} + \frac{G'^2}{4}[M^2(G) + 2M'(G) + 4J_{n,m}] - \frac{(D-1)(D-3)}{4r^2} \tag{9}$$

where the Schwartzian derivative symbol (Hille 1969),

$$\{G,r\} = \frac{G'''(r)}{G'(r)} - \frac{3}{2}\frac{G''^2(r)}{G'^2(r)}.$$

It is noteworthy that the potential given by expression (8) has a term $\frac{(D-1)(D-3)}{r^2}$ which is a constant background attractive inverse square potential in any arbitrary dimension except dimensions 1 and 3, when the generated potential is pure non-power law. We may mention that for the power-law cases, this background potential along with the potential coming from the Schwartzian derivative conspire to give the correct centrifugal barrier potential in arbitrary dimensions (Ahmed 1997).

Equation (9) is the master equation and the subsequent findings of this formalism are based on it. The right-hand side (rhs) of equation (9) is actually a function $B(G, G', G'', G''')$ and is given by

$$B(G, G', G'', G''') = -\frac{1}{2}\{G,r\} - \frac{(D-1)(D-3)}{4r^2} + \frac{G'^2}{4}[M^2(G) + 2M'(G) + 4J_{n,m}]. \tag{10}$$

Table 1. Orthogonal polynomials and their characteristic quantities.

SN	$Q_n^m(G)$	$S(G)$	$W(G)$	$M(G)$	$J_{n,m}$
1	$P_n^{(\alpha,\beta)}(G)$	$1-G^2$	$(1-G)^\alpha(1+G)^\beta$	$\frac{\beta-\alpha-(\alpha+\beta+2)G}{1-G^2}$	$-\frac{n(n+\alpha+\beta+1)}{1-G^2}$
2	$P_n(G)$	$1-G^2$	1	$\frac{-2G}{1-G^2}$	$-\frac{n(n+1)}{1-G^2}$
3	${}_2F_1(\alpha,\beta,\gamma;G)$	$G(1-G)$	$G^\alpha(1-G)^\beta$	$\frac{\gamma-(\alpha+\beta+1)G}{G(1-G)}$	$\frac{\alpha\beta}{G(1-G)}$
4	$L_n^\alpha(G)$	G	$G^\alpha\,\mathrm{e}^{-G}$	$\frac{\alpha+1-G}{G}$	$-\frac{n}{G}$
5	$P_n^m(G)$	$1-G^2$	1	$\frac{-2G}{1-G^2}$	$\frac{n(n+1)}{1-G^2}-\frac{m^2}{(1-G^2)^2}$
6	$C_n^{(\alpha)}(G)$	G^2-1	$(-1)^{\alpha-\frac{1}{2}}(1-G^2)^{\alpha-\frac{1}{2}}$	$\frac{-(2\alpha+1)G}{1-G^2}$	$-\frac{n(n+2\alpha)}{1-G^2}$
7	$T_n(G)$	$1-G^2$	$(1-G^2)^{-\frac{1}{2}}$	$\frac{-G}{1-G^2}$	$-\frac{n^2}{1-G^2}$
8	$H_n(G)$	1	e^{-G^2}	$-2G$	$-2n$

$M(G)$ and $J_{n,m}$ are specified through the differential equation satisfied by the COP $Q_n^m(G)$: $\frac{Q_n^{m\prime\prime}(G)}{Q_n^m(G)}+M(G)\frac{Q_n^{m\prime}(G)}{Q_n^m(G)}+J_{n,m}=0$, $M(G)=\frac{\mathrm{d}}{\mathrm{d}G}\ln S(G)W(G)$.

Expression (9) can be cast in the standard Schrödinger equation form if $B(G,G',G'',G''')$ can be written in the form

$$B(G,G',G'',G''')=E_n-V(r) \tag{11}$$

(say). When $V(r)$ is independent of quantum number n, the Schrödinger equation so obtained would be normal/physical, otherwise it will be Sturmian.

Once a particular COP is chosen to generate an exact analytic solution (EAS), the characteristic functions of the polynomial $M(G)$, $J_{n,m}$ along with the function $G(r)$ get specified. However, at this stage, $B(G,G',G'',G''')$ will not be reduced to the form given by equation (11). To get the energy term in (11) we have to choose one or more than one term(s) in $B(G,G',G'',G''')$ and put it equal to a constant which eventually can be identified with the energy eigenvalues, E_n. If $B(G)$ has k terms, then in practice, we can choose the constant in p numbers of ways, where $p\leqslant(2^k-2)$. This procedure yields a standard Schrödinger equation for expression (8). The calculational scheme can be repeated for other COPs similarly by considering corresponding $M(G)$ and $J_{n,m}$ which are listed in table 1. The construction of ESP will entail finding a potential along with the energy eigenvalues and normalizable eigenfunctions. The procedure is worked out in detail for the case of hypergeometric polynomials.

3. Normalizability of the eigenfunctions of the ESPs

The eigenfunction, in general form, is given by expression (1) where the pre-factor $F(G(r))$ given by equation (6) can be written in terms of $S(G)$, $W(G)$ (see the appendix) as

$$F(G)=NG'^{-\frac{1}{2}}r^{-\frac{D-1}{2}}(W(G)S(G))^{\frac{1}{2}}. \tag{12}$$

This makes the energy eigenfunction become

$$\psi(r)=NG'^{-\frac{1}{2}}r^{-\frac{D-1}{2}}(W(G)S(G))^{\frac{1}{2}}Q_n^m(G). \tag{13}$$

Normalizability of $\psi(r)$ requires

$$\int_{G^{-1}(a)}^{G^{-1}(b)}G'^{-1}r^{-(D-1)}W(G)S(G)\left[Q_n^{m'}(G)\right]^2r^{D-1}\,\mathrm{d}r=\text{finite}=\theta(\text{say}) \tag{14}$$

where $[a, b]$ is the domain of $Q_n^m(r)$.

$$\theta = \int_{G^{-1}(a)}^{G^{-1}(b)} G'^{-1} W(G) S(G) \left[Q_n^{m'}(G)\right]^2 \,\mathrm{d}r$$

and

$$\theta = \int_a^b \frac{S(G)W(G)}{G'^2} \left[Q_n^{m'}(G)\right]^2 \,\mathrm{d}G \tag{15}$$

where m' is a parameter to be determined subsequently for convergence of the integral given by equation (15).

The weight function $W(G)$ and auxiliary function $S(G)$ are characteristics of the particular polynomial considered. Their product is related to $M(G)$ through the relation

$$M(G) = \frac{\mathrm{d}}{\mathrm{d}G} \ln S(G)W(G) \tag{16}$$

where the expressions for $M(G)$ and $W(G)$ of the various COPs are listed in table 1, from which $S(G)$ can be determined. The functional form of $G'^2(r)$ and the characteristic functions $S(G)$ and $W(G)$ contain similar factors of G raised to the powers m_1, m_2, m_3, say, i.e.,

$$\frac{S(G, m_1)W(G, m_2)}{[G'(G, m_3)]^2} = W(G, m). \tag{17}$$

The powers m_1, m_2, m_3 are so chosen that we can recover the weight function $W(G, m)$, with a definite m ($m = m(m_1, m_2, m_3)$) value required for normalization. Equations (15) and (17) then guarantee the finiteness of θ, given by equation (14), once we adjust m' to m as given by equation (44) of the appendix.

4. Construction of ESP when COP is the hypergeometric function

Identifying

$$Q_n^m(G(r)) = {}_2F_1(a, b, c, G) \tag{18}$$

as the hypergeometric function and using the expressions of $M(G)$ and $J_{n,m}$ from table 1 we have

$$M(G) = \frac{c - (a + b + 1)G}{G(1 - G)} \tag{19}$$

and

$$J_{n,m} = \frac{ab}{G(1 - G)}. \tag{20}$$

Equations (9), (18)–(20) yield

$$\begin{aligned}
\frac{\psi''(r)}{\psi(r)} + \frac{D-1}{r}\frac{\psi'(r)}{\psi(r)} &= \frac{c^2G'^2}{4G^2(1-G)^2} + \frac{(a+b+1)(a+b-1)G'^2}{4(1-G)^2} \\
&\quad + \frac{3G''^2}{4G'^2} - \frac{c(a+b)G'^2}{2G(1-G)^2} - \frac{cG'^2}{2G^2(1-G)} - \frac{G'''}{2G'} + \frac{abG'^2}{G(1-G)} \\
&= \left(\frac{c(c-a-b-1)}{2} + ab\right)\frac{G'^2}{G} + \frac{c(c-2)}{4}\frac{G'^2}{G^2} \\
&\quad + \left(\frac{c}{2}(c-a-b-1) + ab\right)\frac{G'^2}{(1-G)} - \frac{1}{2}\left\{\frac{G'''}{G'} - \frac{3G''^2}{2G'^2}\right\} \\
&\quad + \left(\frac{(a+b-c)^2 - 1}{4}\right)\frac{G'^2}{(1-G)^2} - \frac{(D-1)(D-3)}{4r^2}.
\end{aligned} \tag{21}$$

The quantity inside the curly brackets is usually called the Schwartzian derivative and has the symbol $\{G, r\}$. In order to convert equation (21) into a standard stationary state Schrödinger equation, so that the rhs of equation (21) becomes $(E_n - V(r))$, we make one or more terms of the rhs of equation (21) a constant quantity. This, in general, enables us to get the expressions for energy eigenvalues E_n, the functional form of $G(r)$ and subsequently solvable potential $V(r)$. Equation (21) has six terms containing $G(r)$ and its derivatives with respect to r. In order to specify $G(r)$ and to get the energy eigenvalues we have to club together N terms and put it equal to a constant. This can be done, in principle, $\sum_{N=1}^{5} {}^6C_N$ ways, i.e., 62 different ways, discarding the free-particle case. It is to be noted that when $N \geqslant 2$, the analytical computations required to obtain $G(r)$ become increasingly involved.

(i) As a first choice ($N = 1$) let us consider,

$$\frac{G'^2}{G} = p_1^2 \tag{22}$$

where p_1^2 is a real positive constant independent of r. Equation (22) gives the functional form as

$$G(r) = \frac{p_1^2}{4} r^2. \tag{23}$$

Equations (21)–(23) yield

$$\psi''(r) + \frac{D-1}{r}\psi'(r) + (E_a - V(r))\,\psi(r) = 0 \tag{24}$$

where

$$E_a = -p_1^2 \left(a(b+1) + \frac{c(c-a-b-1)}{2} \right) \tag{25}$$

$$V(r) = \left(a(b+1) + \frac{c(c-a-b-1)}{2} + \frac{(a+b-c)^2 - 1}{\left(4 - p_1^2 r^2\right)} \right) \frac{p_1^4 r^2}{\left(4 - p_1^2 r^2\right)} + \left(c(c-2) + \frac{3}{4} - \frac{(D-1)(D-3)}{4} \right) \frac{1}{r^2} \tag{26}$$

and

$$\psi(r) = r^{c-\frac{D}{2}} \left(1 - \frac{p_1^2 r^2}{4} \right)^{\frac{a+b-c+1}{2}} {}_2F_1\left(a, b+1, c; \frac{p_1^2 r^2}{4} \right). \tag{27}$$

The inverse scale factor p_1, for simplicity, is taken as 2 (Ångstrom)$^{-1}$. We take $a = -n$ to make the hypergeometric function a polynomial in r^2, n being a positive integer. Further, to get the correct centrifugal barrier term in D-dimensional Euclidean space, we have to identify the coefficient of r^{-2} in equation (26) as $l(l + D - 2)$ which fixes $c = l + \frac{D}{2}$.

This yields the energy eigenvalues, potential and energy eigenfunction as

$$E_n = \left(4n(b+n) - 2\left(l + \frac{D}{2} \right)\left(l + \frac{D}{2} - b \right) \right) \tag{28}$$

$$V(r) = \frac{r^2}{(1-r^2)} \left(-4n(b+n) + 2\left(l + \frac{D}{2} \right)\left(l + \frac{D}{2} - b \right) \right) + \frac{r^2}{(1-r^2)^2} \left(\left(b - l - \frac{D}{2} - 1 \right)^2 - 1 \right) + \frac{l(l+D-2)}{r^2} \tag{29}$$

and

$$\psi(r) = r^l(1-r^2)^{\frac{b-l-\frac{D}{2}}{2}}\,{}_2F_1\left(-n, b+n, l+\frac{D}{2}; r^2\right), \qquad b-l-\frac{D}{2} \geqslant 0 \tag{30}$$

respectively.

This QS, however, remains Sturmian, as the potential is found to be non-conformable to system specific conversion techniques that sometimes allow the conversion of Sturmian QS to normal/physical QS.

(ii) Continuing the procedure to construct exactly solvable QS we consider the second term of equation (21) to be a constant independent of r, i.e.,

$$\frac{G'^2}{G^2} = p_2^2. \tag{31}$$

This gives the functional form of $G(r)$ as

$$G(r) = A_2 \exp(-p_2 r) \tag{32}$$

A_2 being an integration constant which has to be unity in order to get the correct domain [0, 1] of the hypergeometric polynomial. Equations (21) and (32) yield

$$\psi''(r) + \frac{D-1}{r}\psi'(r) + (E_a - V(r))\psi(r) = 0 \tag{33}$$

where

$$E_a = -\frac{p_2^2(c-2)^2}{4} \tag{34}$$

$$V(r) = \left[\frac{(c-1)(c-a-b-2)}{2} + ab\right]\frac{p_2^2\exp(-p_2 r)}{1-\exp(-p_2 r)} + \frac{(a+b-c+1)^2-1}{4}\frac{p_2^2\exp(-2p_2 r)}{(1-\exp(-p_2 r))^2} - \frac{(D-1)(D-3)}{4r^2} \tag{35}$$

and

$$\psi(r) = r^{-\frac{D-1}{2}}(\exp(-p_2 r(c-2)/2))(1-\exp(-p_2 r))^{\frac{a+b-c+2}{2}}\,{}_2F_1(a, b, c-1, \exp(-p_2 r)). \tag{36}$$

To make ${}_2F_1(a, b, c-1, \exp(-p_2 r))$ a COP we take $a = -n$, where n is a positive integer. Further, the Sturmian potential (35) yields to conversion technique to become normal when

$$a + b - c + 1 = \pm 1. \tag{37}$$

From condition (37) we get the following potentials:
when

$$a + b - c = 0 \tag{38}$$

$$V_1^s(r) = -(n+1)(n-1+c)\frac{p_2^2\exp(-p_2 r)}{1-\exp(-p_2 r)} - \frac{(D-1)(D-3)}{4r^2}. \tag{39}$$

This Sturmian potential can be converted into a normal one, if we attribute to the as yet arbitrary parameter c, an n-dependence such that

$$(n+1)(n-1+c) = \beta_1^2$$

leading to $c = \frac{\beta_1^2+1-n^2}{n+1}$, where β_1^2 is a characteristic constant of the QS independent of n. This leads to the exactly solvable normal potential,

$$V_1(r) = -\frac{\beta_1^2 p_2^2\exp(-p_2 r)}{1-\exp(-p_2 r)} - \frac{(D-1)(D-3)}{4r^2} \tag{40}$$

Table 2. $-[E - V(r)]$ when $Q_n^m(G)$ is ${}_2F_1\,(-n, \beta, \gamma, G)$.

SN	$G(r)$	$\frac{G'^2}{G}$	$\frac{G'^2}{G^2}$	$\frac{G'^2}{(1-G)^2}$	$\frac{G''^2}{G'^2}$	$\frac{G'''}{G'}$	$\frac{G'^2}{1-G}$
1_N	$\frac{p_1^2 r^2}{4}$	p_1^2	$\frac{4}{r^2}$	$\frac{4p_1^2}{(1-p_1^2 r^2)^2}$	r^{-2}	0	$\frac{p_1^4 r^2}{4-p_1^2 r^2}$
2	$\exp(-p_2 r)$	$p_2^2 \exp(-p_2 r)$	p_2^2	$\frac{p_2^2 \exp(-2p_2 r)}{(1-\exp(-p_2 r))^2}$	p_2^2	p_2^2	$\frac{p_2^2 \exp(-2p_2 r)}{1-\exp(-p_2 r)}$
3	$(1-\exp(-p_3 r))$	$\frac{p_3^2 \exp(-2p_3 r)}{1-\exp(-p_3 r)}$	$\frac{p_3^2 \exp(-2p_3 r)}{(1-\exp(-p_3 r))^2}$	p_3^2	p_3^2	p_3^2	$p_3^2 \exp(-p_3 r)$
4	$\exp(-p_4 r)$	$p_4^2 \exp(-p_4 r)$	p_4^2	$\frac{p_4^2 \exp(-2p_4 r)}{(1-\exp(-p_4 r))^2}$	p_4^2	p_4^2	$\frac{p_4^2 \exp(-2p_4 r)}{1-\exp(-p_4 r)}$
5_N	$\sinh p_5 r$	$p_5^2 \cosh p_5 r \coth p_5 r$	$p_5^2 \coth^2 p_5 r$	$\frac{p_5^2 \cosh^2 p_5 r}{(1-\sinh p_5 r)^2}$	$p_5^2 \tanh^2 p_5 r$	p_5^2	$\frac{p_5^2 \cosh p_5 r}{1-\sinh p_5 r}$
6_N	$p_6 r - \frac{p_6^2 r^2}{4}$	$\frac{p_6(1-\frac{p_6 r}{2})^2}{r(1-\frac{p_6^2 r}{4})}$	$\frac{(1-p_6 r)^2}{(1-\frac{p_6 r}{4})^2}$	$\frac{p_6^2}{(1-\frac{p_6 r}{2})^2}$	$\frac{p_6^2}{4(1-\frac{p_6 r}{2})^2}$	0	p_6^2

Term(s) chosen as constant(s), p_i in the rhs of equation (21) whose sum in the jth row gives the energy eigenvalue $(-E_n)_j$ and sum of the r-dependent quantities, except for $G(r)$, gives ESP $[V(r)]_j$ of the jth EQS when multiplied by appropriate constants in equation (21). Subscript N in SN indicates new ESP.

along with the energy eigenvalues:

$$E_n = -p_2^2 \left(\frac{\beta_1^2 - (n+1)^2}{2(n+1)} \right)^2 \tag{41}$$

$\beta_1^2 > (n+1)^2$ and the energy eigenfunction:

$$\psi(r) = r^{-\frac{D-1}{2}} (\exp(-p_2 r))^{\frac{\beta_1^2 - (n+1)^2}{2(n+1)}} \times (1 - \exp(-p_2 r))\, {}_2F_1 \left(-n, \frac{\beta_1^2 + n + 1}{n+1}, \frac{\beta_1^2 - n^2 - n}{n+1}; \exp(-p_2 r) \right). \tag{42}$$

When $a + b - c = -2$, we get a potential similar to (40) except that the c parameter has a definite n-dependence[3].

Other possibilities for fixing c have been explored and it has been found that no other normal ESPs can be obtained from $V(r)$ (equation (35)). Table 2 summarizes the consequences of choosing different terms in the rhs of equation (21) to represent the energy eigenvalue when the COP used is ${}_2F_1(-n, \beta, \gamma; G)$. In a similar way, by identifying $Q_n^m(G)$ as another COP, mentioned in table 1 and applying the above mentioned procedure different ESPs can be obtained. To avoid repetitions we present the summaries of the consequences of choosing the different terms as constants when the COPs are associated Laguerre and associated Legendre in tables 3 and 4. Figure 1 contains graphs of some new ESPs along with their corresponding energy eigenfunctions in arbitrary units profiling their behaviour in r. These graphs are illustrative and not exhaustive.

5. Discussion

In this paper, we have presented a method of construction/generation of ESPs of the Schrödinger equation based on the properties of COPs. In fact this method is a refinement of the method presented in paper I, as it is more flexible than the former method, in the sense that within the context of a particular COP, this method admits the generation of more ESPs.

[3] As there are no constraint equations connecting the constants in equations (40)–(42), they could have been written in a much simpler form by making the constants unity. But we desist from making the expressions simpler looking as there is a hidden flexibility in this complexity which will be apparent at the time of application.

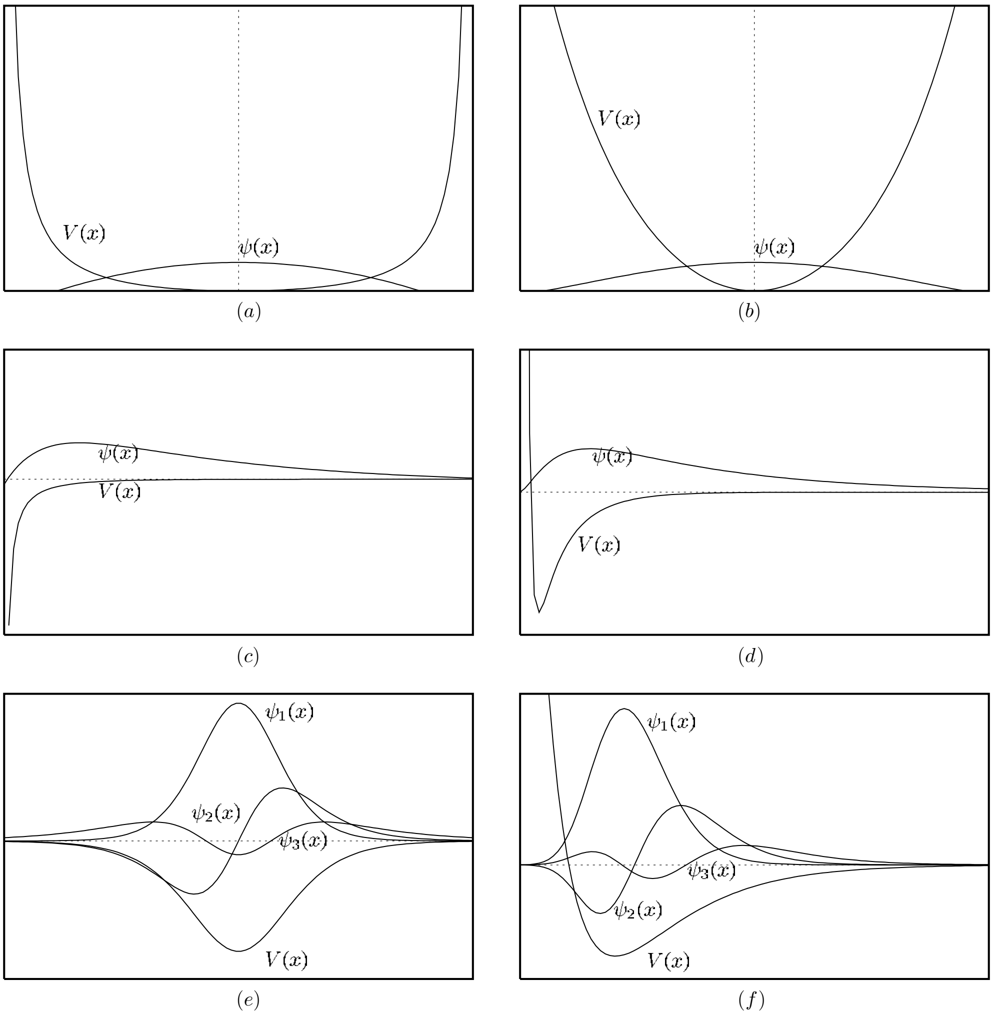

Figure 1. The ESPs $V(x)$ and energy eigenfunctions $\psi(x)$ for Sturmian EQSs ((*a*) and (*b*)) and ((*c*) and (*d*)), derived from ${}_2F_1(-n, \beta, \gamma; G)$ (SN 1, table 2) and $P_n^m(G)$ (SN 4, table 4), respectively. The normal ESPs $V(x)$ with ground-state energy eigenfunction $\psi_1(x)$ and excited-state energy eigenfunctions $\psi_2(x)$, $\psi_3(x)$ ((*e*) and (*f*)) derived from $P_n^m(G)$ (SN 2, table 4) and $L_n^\alpha(G)$ (SN 1, table 3), respectively. All graphs are drawn in arbitrary units.

The method is applied to generate spherically symmetric ESPs in an arbitrary *D*-dimensional Euclidean space. ESPs so generated are generally Sturmian. However, as shown, some of the Sturmian ESPs yield to system specific conversion techniques and become normal/physical ESPs. The number of possible ESPs that can be generated for a COP depends on the number of terms in $B(G, G', G'', G''')$ and the mode of extraction of the energy eigenvalues as illustrated in the text for hypergeometric polynomials. Construction of ESPs for other COPs, namely associated Laguerre and associated Legendre, is also similar to the case as elaborated for a hypergeometric polynomial. Their construction procedure is summarized in tables 3 and 4. In tables 3 and 4, except for $G(r)$, the other quantities in the first row are terms of $\Omega(r)$ containing $G(r)$ and its higher order derivatives when COP is associated Laguerre and associated Legendre, respectively. As per our procedure one of the terms of $\Omega(r)$ is put equal

Table 3. $-[E - V(r)]$ when $Q_n^m(G)$ is associated Laguerre ($L_n^\alpha(G)$).

SN	$G(r)$	$\frac{G'^2}{G^2}$	G'^2	$\frac{G'^2}{G}$	$\frac{G''^2}{G'^2}$	$\frac{G'''}{G'}$
1_N	$\exp(-p_1 r)$	p_1^2	$p_1^2 \exp(-2p_1 r)$	$p_1^2 \exp(-p_1 r)$	p_1^2	p_1^2
2	$p_2 r$	$\frac{1}{r^2}$	p_2^2	$\frac{p_2}{r}$	0	0
3	$\frac{p_3^2}{4} r^2$	$\frac{4}{r^2}$	$\frac{p_3^4}{4} r^2$	p_3^2	$\frac{1}{r^2}$	0
4	$\exp(-p_4 r)$	p_4^2	$p_4^2 \exp(-2p_4 r)$	$p_4^2 \exp(-p_4 r)$	p_4^2	p_4^2
5_N	$\sinh p_5 r$	$p_5^2 \coth^2 p_5 r$	$p_5^2 \cosh^2 p_5 r$	$p_5^2 \cosh p_5 r \coth p_5 r$	$p_5^2 \tanh^2 p_5 r$	p_5^2

Term(s) taken as constant(s), p_i whose sum in the jth row gives the energy eigenvalue $(-E_n)_j$ and sum of the r-dependent quantities, except for $G(r)$, gives ESP $[V(r)]_j$ of the jth EQS when multiplied by appropriate constants in the corresponding equation for $L_n^\alpha(G)$ similar to equation (21). Subscript N in SN indicates new ESP.

Table 4. $-[E - V(r)]$ when $Q_n^m(G)$ is associated Legendre ($P_n^m(G)$).

SN	$G(r)$	$\frac{G'^2}{1-G^2}$	$\frac{G'^2}{(1-G^2)^2}$	$\frac{G^2 G'^2}{(1-G^2)^2}$	$\frac{G''^2}{G'^2}$
1	$\sin p_1 r$	p_1^2	$p_1^2 \sec^2 p_1 r$	$p_1^2 \tan^2 p_1 r$	$p_1^2 \tan^2 p_2 r$
2	$\tanh p_2 r$	$p_2^2 sech^2 p_2 r$	p_2^2	$p_2^2 \tanh^2 p_2 r$	$p_2^2 \tanh^2 p_2 r$
3_N	$(1 - \exp(-2p_3 r))^{\frac{1}{2}}$	$\frac{p_3^2 \exp(-2p_3 r)}{1-\exp(-2p_3 r)}$	$\frac{p_3^2}{1-\exp(-2p_3 r)}$	p_3^2	$\frac{p_3^2 (2-\exp(-2p_3 r))^2}{(1-\exp(-2p_3 r))^2}$
4_N	$\exp(-p_4 r)$	$\frac{p_4^2 \exp(-2p_4 r)}{1-\exp(-2p_4 r)}$	$\frac{p_4^2 \exp(-2p_4 r)}{(1-\exp(-2p_4 r))^2}$	$\frac{p_4^2 \exp(-4p_4 r)}{(1-\exp(-2p_4 r))^2}$	p_4^2

Term(s) taken as constant(s) p_i whose sum in the jth row gives the energy eigenvalue $(-E_n)_j$ and sum of the r-dependent quantities, except for $G(r)$, gives ESP $[V(r)]_j$ of the jth EQS when multiplied by appropriate constants in the corresponding equation for $P_n^m(G)$ similar to equation (21). Subscript N in SN indicates new ESP.

to a constant which eventually becomes the energy eigenvalue. This step yields the analytical form of $G(r)$ making other terms of $\Omega(r)$ specified, allowing us to write $\Omega(r) = V(r) - E_n$. In table 3, $\Omega(r)$ contains five terms and hence there are five simple ways to choose the constant term. However, it is also possible to choose two or more terms as constant. This opens the possibilities of having $\sum_{m=1}^{4} {}^5C_m(=30)$ different choices of choosing E_n. Similar explanations hold for table 4. In this method normalizability of the eigenfunction for an ESP plays an important role in shaping the final form of the generated ESP. As explained in the text by fixing the index m of a chosen $Q_n^m(G)$ from normalizability consideration of the energy eigenfunction, it is possible to get a number of normal ESPs. The ESPs generated are mostly non-power law with an inverse square potential $(D-1)(D-3)r^{-2}$ which vanishes for $D = 1$ or 3. It may be mentioned that this term along with the Schwartzian derivative gives the correct form of the D-dimensional 'centrifugal barrier' term for a power-law potential. Further, it may be noted that we have kept the various constants such as integration constants, scale factors, 'characteristic constants' explicitly in our expressions instead of putting them equal to unity to make them simpler. This apparent complexity, however, allows flexibility to the generated ESPs at the time of possible applications.

Appendix

We present here briefly the relevant information regarding the COPs.

If $Q_n^m(r)$ represents an nth degree polynomial in the variable r defined through the generalized Rodrigues formula (Dennery and Krzywicki 1969)

$$Q_n(r) = \frac{1}{W}\frac{\mathrm{d}^n}{\mathrm{d}r^n}(W(r)(S(r))^n) \tag{43}$$

with the help of the weight $W(r)$ and an auxiliary function $S(r)$; then $\{Q_n(r)\}(n = 0, 1, 2, 3\ldots)$ would form an orthogonal set with weight $W(r)$ on the interval $[a, b]$ provided $Q_n(r)$, $W(r)$ and $S(r)$ satisfy

(i) $Q_n(r)$ is a first degree polynomial in r;
(ii) $S(r)$ is a polynomial in r of degree $\leqslant 2$, with real roots of the equation, $S(r) = 0$;
(iii) $W(r)$ is positive, real and integrable in the domain $[a, b]$ and the combination $W(r)S(r)$ satisfies the boundary condition

$$W(a)S(a) = 0 = W(b)S(b).$$

Further, $Q_n(r)$ will be called a COP and would satisfy the differential equation:

$$(S(r)W(r)Q_n'(r))' = -\lambda_n W(r)Q_n(r)$$

where a prime represents differentiation with respect to its argument and the constant λ_n is given by

$$\lambda_n = -n\left[K_1 Q_1'(r) + \tfrac{1}{2}(n-1)S''(r)\right]$$

K_1 being a constant utilized for the standardization of the polynomials[4]. The corresponding associated orthogonal polynomials defined over the same domain $[a, b]$ are obtained (Jafarizadeh and Fakhri 1996) through the relation (except for Hermite and Legendre)

$$Q_n^m(r) = (-1)^m S(r)^{\frac{m}{2}} \frac{\mathrm{d}^m}{\mathrm{d}r^m} Q_n(r)$$

where m is at most n and satisfies the following second-order differential equation:

$$\left(W(r)S(r)Q_n^{m'}(r)\right)' = -\eta_{n,m} W(r)Q_n^{m'}(r)$$

where

$$\eta_{n,m} = \left[\frac{1}{2}(m^2n^2 - 2m + n)\frac{S''}{S} + \frac{m-n}{S}\left(\frac{(SW)'}{W}\right)' - \frac{m^2}{4}\frac{S'^2}{S^2} - \frac{m}{2}\frac{W'}{W}\frac{S'}{S}\right].$$

Further, $Q_n^m(r)$ satisfies the normalization condition:

$$\int_a^b W(r)\left[Q_n^m(r)\right]^2 \mathrm{d}r = N(n, m) = \text{finite}. \tag{44}$$

References

Ahmed S A S 1997 *Int. J. Theor. Phys.* **36** 1893
Ahmed S A S and Borah B C 2002 *Indian J. Phys.* B **76** 205
Ahmed S A S, Borah B C and Sharma D 2001 *Eur. Phys. J.* D **17** 5
Biswas S N, Dutta K, Saxena R P, Srivastava P K and Verma V S 1971 *Phys. Rev.* D **4** 3617
Bose S K 1994 *Nuovo Cimento* B **109** 311
Chhajlany S C and Malnev V N 1990 *J. Phys. A: Math. Gen.* **23** 3711
Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
Dennery P and Krzywicki A 1969 *Mathematics for Physicist* (New York: Harper and Row)
Dutra A de Souza 1988 *Phys. Lett.* A **131** 319
Dutra A de Souza 1993 *Phys. Rev.* A **47** R2435
Dutra A de Souza and Filho H Boschi 1991 *Phys. Rev.* A **44** 4721
Dutta R, Khare A and Varshni Y P 1995a *J. Phys. A: Math. Gen.* **28** L107
Dutta R, Varshni Y P and Adhikari B 1995b *Mod. Phys. Lett.* A **10** 597
Flessas G P 1979 *Phys. Lett.* A **72** 289
Flessas G P and Das K P 1980 *Phys. Lett.* A **78** 19

[4] $K_1 = (-1)^n$ for Hermite, $(-2)^n n!$ for Jacobi and Legendre, $n!$ for Laguerre, etc.

Hille E 1969 *Lectures on Ordinary Differential Equations* (Reading, MA: Addison-Wesley) p 647
Jafarizadeh M A and Fakhri H 1996 *Indian J. Phys.* B **70** 465
Khare A 1981 *Phys. Lett.* A **83** 237
Roy P and Roychoudhury R 1987 *J. Phys. A: Math. Gen.* **20** 6597
Manning M F 1935 *Phys. Rev.* **48** 161
Nieto M M 1984 *Phys. Lett.* B **145** 208
Shiffman M A 1989 *Int. J. Mod. Phys.* A **4** 2897